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On Some Nonlinear Control System Problems on a Finite Time Interval. ⋆

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Abstract: In this presentation we evaluate nonlinear control systems on a finite time interval. We also discuss application of the invariance features and weak invariant sets in the procedures of constructing solutions for approach problems. These problems are closely connected with evaluation of reachability sets and integral funnels of control systems. We revise several problems of approaching and one general scheme of constructing solutions for these problems. Scheme consist of two stages: at the first stage we construct approximate reachability set of the approach problem; at the second stage in one way or another we construct control on a given time interval that solves the approach problem.

Keywords: optimal control, dynamic system, approximation, reachability sets, attainability sets

1. INTRODUCTION

In this presentation we evaluate nonlinear control systems on a finite time interval. Specifically we consider problems of approaching with compact target set in a phase space [1-5].

We also discuss application of the invariance features and weak invariant sets in the procedures of constructing solutions for approach problems. These problems are closely connected with evaluation of reachability sets and integral funnels of control systems. Practice problems demand the proper approaches for solving these problems. At the moment there are some effective approaches for solving problem of integral funnels and reachability sets approximation. Here we should mention approach based on ellipsoidal estimations of reachability sets [4,6,7,8] as well as the approach based on polyhedral approximations [10]. Pixel method is one of the approaches based on polyhedral approximations. Advantage of the pixel method is that its constructions are simple and easy to work with, in particular, it is easy to construct union and intersection of pixel sets. Another advantage of the pixel method is the possibility of its application to the high-dimension problems. These algorithms are widely used not only for algorithms development in the control theory but also for other problems in different branches of mathematics, e.g. the image recognition theory.

In this presentation we revise several problems of approaching and one general scheme of constructing solutions for these problems. Scheme consist of two stages: at the first stage we construct approximate reachability set of the approach problem; at the second stage in one way or another we construct control on a given time interval that solves the approach problem. Algorithms of constructing reachability sets and of solving control utilize time interval discretisation and some kind of control system phase space discretisation, this way algorithms allows to obtain approximate solution for approach problem. Presentation considers several examples of named algorithms application for solving specific appproach problems for nonlinear control systems. Present work is based on works [1-13].

2. CONTROL SYSTEMS AND DIFFERENTIAL INCLUSIONS, INVARIANT SETS

On the time interval $[t_0, \vartheta]$ the following system is given:

$$\frac{dx}{dt} = f(t, x, u), x \in \mathbb{R}^n; \tag{1}$$

here u — control vector, where

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$$u \in P, \tag{2}$$

is a compact in Euclidean space $\mathbb{R}^p$.

Controlled system (1), (2) satisfies conditions:

Condition A. Vector-function $f(t, x, u)$ is defined and continuous by t, x, u, and for any closed and limited domain $D \subset [t_0, \vartheta] \times \mathbb{R}^n$ there exists such constant $L = L(D) \in (0, \infty)$, that

$$\|f(t, x^{(1)}, u) - f(t, x^{(2)}, u)\| \leqslant L\|x^{(1)} - x^{(2)}\|,$$
$$(t, x^{(i)}, u) \in D \times P, \quad i = 1, 2. \tag{3}$$

Condition B. There exists such constant $\gamma \in (0, \infty)$, that

$$\|f(t, x, u)\| \leqslant \gamma(1 + \|x\|),$$
$$(t, x, u) \in [t_0, \vartheta] \times \mathbb{R}^n \times P. \tag{4}$$

Here $\|f\|$ — norm of vector f in Euclidean space.

Assume $\mathcal{F}(t, x) = \{f(t, x, u) : u \in P\}$, $F(t, x) = \mathrm{co}\,\mathcal{F}(t, x)$, $(t, x) \in [t_0, \vartheta] \times \mathbb{R}^n$; here $\mathrm{co}\{f\}$ —convex hull of the set $\{f\}$ of vectors f.

Along with the system (1) in the presentation we consider differential inclusion (d.i.)

$$\frac{dx}{dt} \in F(t, x). \tag{5}$$

Let's introduce some notions. Let $t_0 \leq t_* < t^* \leq \vartheta$ and $x_* \in \mathbb{R}^n$.

With symbol $X(t^*, t_*, x_*)$ we denote reachability set at the time moment t^* (1) from position (t_*, x_*); $X(t^*, t_*, X_*) = \bigcup\limits_{x_* \in X_*} X(t^*, t_*, x_*)$ — reachability set at the time moment t^* from the initial set X_*.

Set $X(t_*, x_*) = \bigcup\limits_{t^* \in [t_*, \vartheta]} (t^*, X(t^*, t_*, x_*)) \subset [t_*, \vartheta] \times \mathbb{R}^n$ we name the integral funnel of the system (1) on the time interval $[t_*, \vartheta]$, satisfying initial position (t_*, x_*); here $(t^*, X^*) = \{(t^*, x^*) : x^* \in X^*\}$.

We also assume $X(t_*, X_*) = \bigcup\limits_{x_* \in X_*} X(t_*, x_*)$ — integral funnel of the system (1) on $[t_*, \vartheta]$, corresponding to the initial pair (t_*, X_*).

By analogy for d.i. (5) we define reachability sets $Y(t^*, t_*, x_*)$, $Y(t^*, t_*, X_*)$ and integral funnels on $[t_*, \vartheta]$.

We denote with $\mathrm{comp}(\mathbb{R}^n)$ metric space, elements of which are compacts from $\mathbb{R}^n$ with Hausdorf metric; we denote with $d(F_*, F^*)$, where F_* F^* from $\mathrm{comp}(\mathbb{R}^n)$.

Let $X_* \in \mathrm{comp}(\mathbb{R}^n)$. From conditions (A) and (B) it follows, that $Y(t^*) = Y(t^*, t_*, X_*) \in \mathrm{comp}(\mathbb{R}^n)$ for $t_0 \leqslant t_* < t^* \leqslant \vartheta$ and the mapping $(t^*, t_*, X_*) \mapsto Y(t^*, t_*, X_*)$ is continuous in Hausdorf metric.

Let us revise some known notions of invariance and weak invariance.

Here Φ — non-empty closed set in $[t_0, \vartheta] \times \mathbb{R}^n$.

Set Φ is invariant regarding to (1), if for any $(t_*, x_*) \in \Phi$ it is following that $X(t_*, x_*) \subset \Phi$.

Set Φ is invariant regarding to (1), if for any $(t_*, x_*) \in \Phi$ there exists solution $x(t)$, $x(t_*) = x_*$ of the system (1), satisfying inclusion $(t, x(t)) \in \Phi$, $t \in [t_*, \vartheta]$.

3. APPROACH PROBLEM FOR THE SYSTEM (1)

Let us formulate several approach problems for the system(1) with the target set in $\mathbb{R}^n$.

In addition to conditions (A) and (B) we assume, that system (1) satisfies condition

Condition C. $\mathcal{F}(t, x) = F(t, x)$ for $(t, x) \in [t_0, \vartheta] \times \mathbb{R}^n$.

Let us consider approach problems in the presence of conditions (A), (B), (C) applied to the system (1).

Let $M \in \mathrm{comp}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$.

Taking into account specificity of the problems we name the solutions of the system (1) (d.i. (5)) trajectories.

Problem 1. It is necessary to find permissible control $u^*(t)$ on $[t_0, \vartheta]$, generating trajectory $x^*(t)$, $x^*(t_0) = x_0$ of the system (1), and satisfying condition $x^*(\vartheta) \in M$.

Considering conditions applied to the system (1),we can find limited and closed domain $D \subset [t_0, \vartheta] \times \mathbb{R}^n$, that contains all that initial positions (t_*, x_*), starting from which it is possible to solve Problem 1. Furthermore, D is invariant set regarding to (1). Such domain D along with the constants $L = L(D)$ $K = K(D)$ we consider in our formulas.

Solving set $W \subset [t_0, \vartheta] \times \mathbb{R}^n$ is the set of all positions $(t_*, x_*) \in D$, starting from which it is possible to solve Problem 1. This set can be described as integral funnel of some control system on $[t_0, \vartheta]$, via turning back the time t.

Namely, along with "straight" time $t \in [t_0, \vartheta]$ we introduce so-called "backward" time $\tau = t_0 + \vartheta - t \in [t_0, \vartheta]$.

We correspond to the system (1) following control system in $\mathbb{R}^n$:

$$\frac{dz}{d\tau} = f^o(\tau, z, v), \tau \in [t_0, \vartheta], \tag{6}$$

that satisfies "backward" time τ; here $f^o(\tau, z, v) = -f(t_0 + \vartheta - t, z, v)$, $(\tau, z, v) \in [t_0, \vartheta] \times \mathbb{R}^n \times P$.

Solving set W of the set (1) is defined by $W(t) = Z(\tau)$, $\tau = t_0 + \vartheta - t$, $\tau \in [t_0, \vartheta]$; here $Z(\tau)$ are the sections of integral funnel $Z = Z(t_0, M) \subset [t_0, \vartheta] \times \mathbb{R}^n$ of the system (6), $W(t)$ are the sections of the set W at the moment t . Set W is weak invariant in relation to the system (1.1)

In general case sets Z can be calculated only approximately . Theme of calculating and estimation of integral funnel are described in works []. For example, description of one the scheme of approximate integral funnel calculation is given in []. Along with the approximately calculated set Z we obtain approximately calculated set W.

In addition to the Problem 1 we also consider following problems.

Problem 2. (Problem of approaching by the moment ϑ). It is necessary for given $x_0 \in \mathbb{R}^n$ and $M \in \mathrm{comp}(\mathbb{R}^n)$ to find admissible control $u^*(t)$ on $[t_0, \vartheta]$, generating trajectory $x^*(t)$, $x^*(t_0) = x_0$ of the system (1), satisfying inclusion $x^*(t^*) \in M$ for some $t^* \in [t_0, \vartheta]$.

Let there are given $M \in \mathrm{comp}(\mathbb{R}^n)$, nonempty set $\Omega \in \mathrm{comp}(\mathbb{R}^n)$ $(M \cap \Omega \neq \varnothing)$ and $x_0 \in \Omega$.

Let us formulate the problem of approaching (1) with M in the presence of phase restriction Ω of the system (1).

Problem 3. It is necessary to find admissible control $u^*(t)$ on $[t_0, \vartheta]$, generating trajectory $x^*(t)$, $x^*(t_0) = x_0$ of the system (1), satisfying inclusions $x^*(t) \in \Omega$, $t^* \in [t_0, \vartheta]$ and $x^*(\vartheta) \in M$.

Problem 4. It is necessary to find admissible control $u^*(t)$ on $[t_0, \vartheta]$, generating trajectory $x^*(t)$, $x^*(t_0) = x_0$ of the system (1),satisfying inclusions $x^*(t^*) \in M$ for some $t^* \in [t_0, \vartheta]$ and $x^*(t) \in \Omega$ for $t \in [t_0, t^*]$.

For problems 2–4 we use the same ideas of constructing approximate reachability sets $W \subset [t_0, \vartheta] \times \mathbb{R}^n$, that we use for Problem 1; at the same time for the algorithms of approximate calculations of the sets W we find some specificity that follows from specificity of the problem.

4. ON CONSTRUCTING SOLVING CONTROLS FOR APPROACH PROBLEMS

At the moment there are known several schemes which are used for constructing admissible solving controls providing approximate solution for approach problems. These schemes utilize discretization of the time interval $[t_0, \vartheta]$ and of the phase space $\mathbb{R}^n$ of the control system (1).

Some of these schemes are based on extreme shift principle of the system (1) trajectory to the set W (see. [1]). The scheme based on maximum attraction of the system (1) trajectory to the set W also possess good results (see. [14], p.). Construction of solutions for a number of mechanical control problems is implemented basing on these schemes. Let us provide typical scheme of solving mechanical control problem.

Example 1.

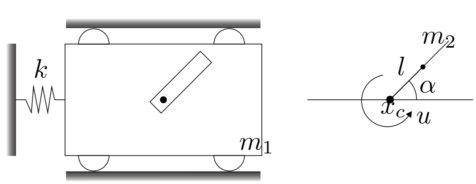


Fig. 1. Translational oscillator with rotating actuator (TORA)

The translational oscillator with rotating actuator (TORA) is a system which contains the platform attached with the spring to the fixed wall. The platform can move only in the horizontal plane along the axis x_1 parallel to the spring. The eccentric which can be rotated by the rotor is placed on the platform. The rotating eccentric creates a driving force which is used to damp the translational motion of the platform. The mass of platform is m_1, the eccentric mass is m_2, the length of eccentric is l, moment of inertial of the eccentric is I, the spring stiffness is linear and equal to k, the control u is the rotor torque moment.

TORA is described with following system of equations

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \dfrac{(m_1 + m_2)u - m_2 l \cos x_1 (m_2 l x_2^2 \sin x_1 - k x_3)}{(I + m_2 l^2)(m_1 + m_2) - m_2^2 l^2 \cos^2 x_1}, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \dfrac{(I + m_2 l^2)(m_2 l x_2^2 \sin x_1 - k x_3) - m_2 l \cos x_1 u}{(I + m_2 l^2)(m_1 + m_2) - m_2^2 l^2 \cos^2 x_1}, \end{cases} \tag{7}$$

here $x_1 = x_c$, $x_3 = \alpha$.

The dynamical system (7) was calculated using next parameters

- The system parameters are next: $m_1 = 1.0$, $m_2 = 0.1$, $l = 1$, $k = 0.1$, $I = 1.2$.
- Time interval $[t_0, \vartheta] = [0, 5]$.
- Initial set $X_0 = \{x \in \mathbb{R}^4 \colon \|x\|_\infty \leqslant 1\}$.
- Time partitions diameter $\Delta(\Gamma) = 0.01$.
- The admissible control set $P = \{u \colon u \in [-1; 1]\}$, $\tilde{P} = \{-1, -0.5, 0, 0.5, 1\}$.
- The point $x_f = (0.28, -0.33, -0.35, -0.52)$.

The projections of 4-dimensional solvability set of the system (7) at the moment $t = 1.25$ calculated with parameters described above are presented on Figure 2: a) projection on x_1, x_2, x_3; b) projection on x_2, x_3, x_4.

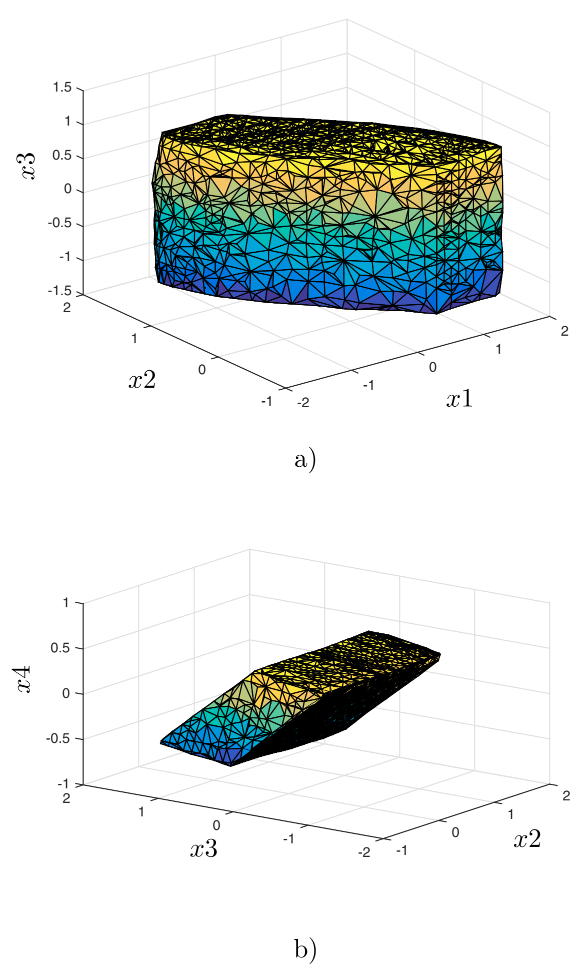
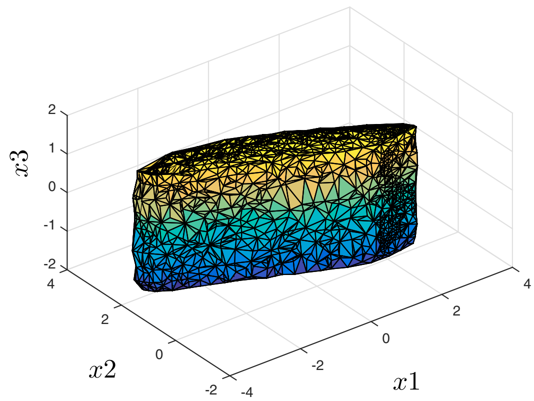
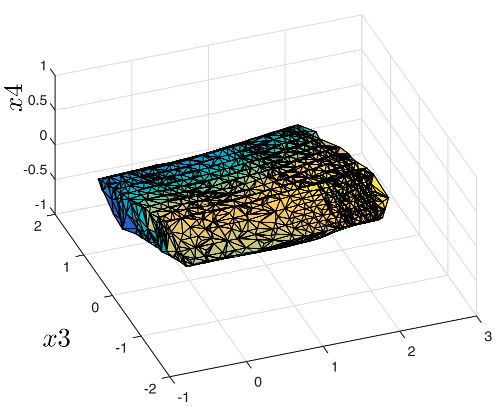


Fig. 2. The projections of 4-dimensional solvability set of the system (7) at the moment $t = 1.25$

The projections of 4-dimensional solvability set of the system (7) at the moment $t = 2.5$ calculated with parameters described above are presented on Figure 3: a) projection on x_1, x_2, x_3; b) projection on x_2, x_3, x_4.



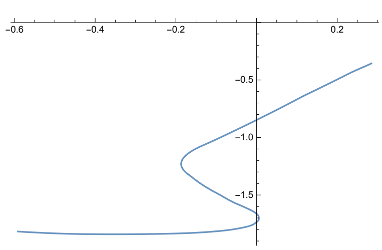
a)



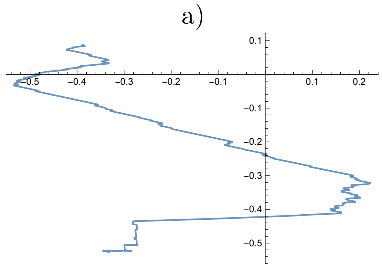
b)

Fig. 3. The projections of 4-dimensional solvability set of the system (7) at the moment $t = 2.5$

Fig. 4-6 show graphs for motion, control and velocity vectors.



a)



b)

Fig. 4. Projections of trajectory (7) on the time interval $[0, 5]$: a) trajectory; b) velocity

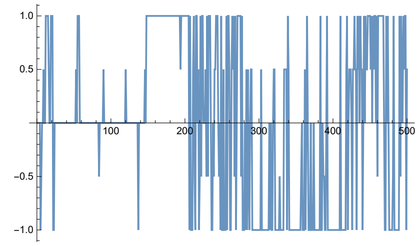


Fig. 5. Control vector

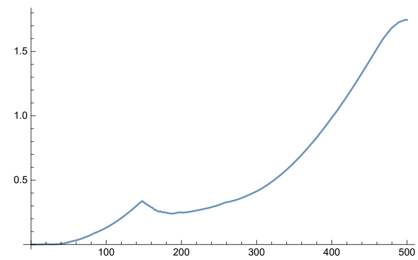


Fig. 6. Error of calculations by attraction method

Example 2. Let it is given nonlinear control mechanical system — reverse pendulum mounted on the train moving along horizontal plane

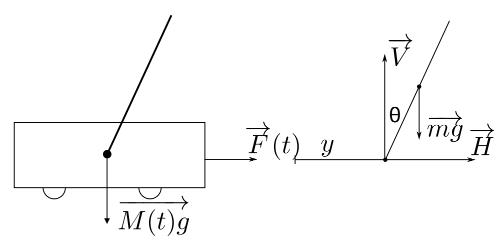


Fig. 7. Train-pendulum mechanical system

Train is affected by force F along horizontal direction. Pendulum is affected by gravity force gravity force mg attached to the gravity center and normal force consisted of horizontal force H and vertical force V; m is the mass of the pendulum and g is the gravity constant.

Let us denote with L the distance between the gravity center of the pendulum and its ground point, y — offset of the pendulum ground point, φ — the angle of pendulum inclination in relation to vertical axe, I — inertia moment of the pendulum in relation to center mass.

Let us introduce variables $x_1 = \varphi$, $x_2 = \dot{\varphi}$, $x_3 = y$, $x_4 = \dot{y}$ and denote with $u(t)$ thrust force of $F(t)$, $t \in [0, \vartheta]$, where $|F(t) \leqslant \mu|$, μ — given positive number. With $k = const$ we denote friction coefficient of horizontal plane.

Equation of mechanical system "train - pendulum" has a following form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{1}{\Delta(x_1)}((M(t)+m)mgL\sin x_1 - \\ -mL\cos x_1(u(t) + mLx_2^2 \sin x_1 - kx_4)) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\dfrac{1}{\Delta(x_1)}(-mL\cos x_1 mgL\sin x_1 + \\ +(I+mL^2)(u(t) + mLx_2^2 \sin x_1 - kx_4); \end{cases} \quad (8)$$

here $\Delta(x_1) = (I + mL^2)(M(t) + m) - m^2L^2 \cos x_1$.

We consider problem of approaching for mechanical system "train - pendulum" in relation for the phase constraint $\Phi \subset R^4$ and with specific parameter values of the system. We consider the problem of approaching in the time moment ν for the system "train - pendulum" from initial point $x^{(0)} \in R^4$. Target of approaching is the single-point set $M = x^f \subset R^4$, here $x^{(0)}$ and $x^{(f)}$ — given points in R^4.

According to the scheme giving approximate solution of the approach problem, we introduce partition $\Gamma = \{0, t_1, t_2, \dots, t_{N-1}, t_n = \nu\}$ of the time interval $[t_0, \nu]$ and for this partition Γ we perform calculations of sequence $\{\widetilde{\mathcal{W}}(t_j)\}$ of the finite sets $\{\widetilde{\mathcal{W}}(t_j)\} \subset R^4, j = N, N-1, \dots, 0, \mathcal{W}_{(t_N)} = \{x^f\}$. These sets approximates resolvability sets W in the approach problem under consideration.

Initial point $x^{(0)}$ is chosen such that $x^{(0)} \in \widetilde{\mathcal{W}}(t_0)$. For this point we calculate solving program control $u^*(t), t \in [0, \nu]$. Control $u^*(t), t \in [0, \nu]$ brings trajectory $x^*(t)$ in the time moment ν to some close neighbourhood of the point x^f. Control $u^*(t), t \in [0, \nu]$ brings trajectory $x^*(t)$ of the system "train-pendulum" to some small neighbourhood of the point x^f.

Let us provide graphic solution of the approach problem in presence of a phase constraint. First we provide specific values of the system parameters.

We consider the system on the time interval $[t_0, \nu] = [0, 2]$ and scalar control $u = u^*(t)$ on $[0, 2]$ is constrained by $u^*(t) \in P = [-1, 1]$. We chose single-point set $M = x^f \subset \mathcal{R}^4$, where $x^f = (0, 0, 1, 0)$ as the target set. For control system "train-pendulum" we chose following values of parameters: $m = 0.25$ kg; $m^* = 2$ kg; $g = 9.81$ m / s^2; $k = 1$; $L = 1$ m; $I = 1$ kg $\cdot$ m^2.

Phase constraint Φ is closed ball in R^4 with the center in the point $O = (0, 0, 1.35, 0)$ and with the diameter $d = 0.1$ m.

Let us provide graphic description of the approach problem solution. Fig. 8,9 presents projections of the sets $\widetilde{\mathcal{M}}(t_j)$ $t_j = 0; 0.5; 1; 1.5; 2$ and of the set Φ on the 3-dimensional subspace $R_{1,2,3}$ of the variables x_1, x_2, x_3 of the space R^4. On the fig. 10 the point x^f is the projection of the target set M on the subspace $R_{1,2,3}$.

On the fig. 12 the scalar control $u^*(t)$ on $[0, 2]$ is shown. This control approximately solves the problem of approaching with M from the point $x^{(0)} =$ $(-0.0016; 0.0017; 1.0131; -0.0360) \in \widetilde{\mathcal{W}}(0)$

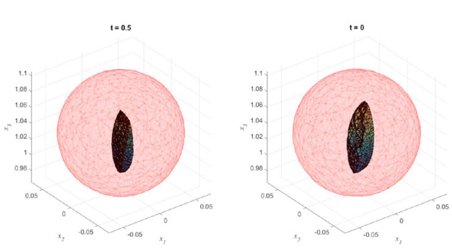


Fig. 8.

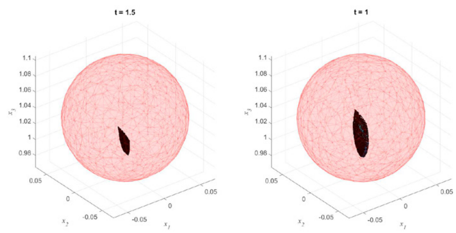


Fig. 9.

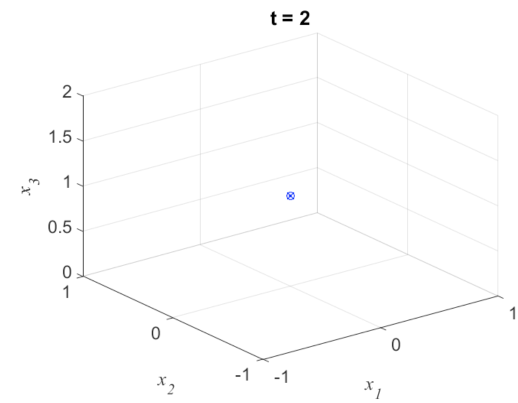


Fig. 10.

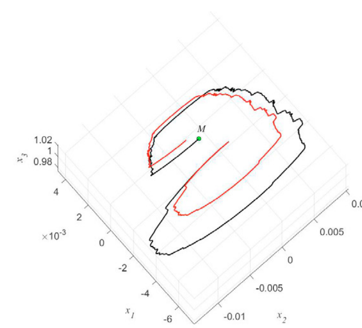


Fig. 11. Control vector $u^*(t)$

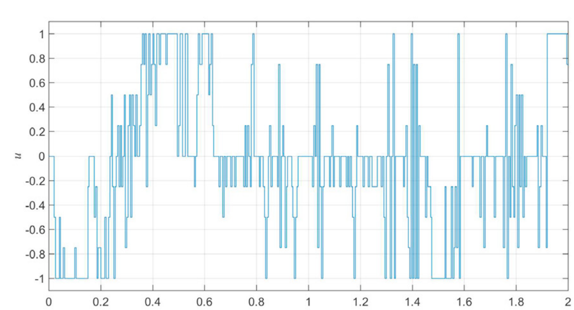


Fig. 12. Trajectory $x^*(t)$

5. CONCLUSION

Current paper consider several typical approach problems for nonlinear control system on a finite time interval. Scheme of solving these problems is given in a brief form as well as several examples of its application for specific mechanical problems.

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